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Non-constant positive steady state of one resource and two consumers model with diffusion ☆

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Abstract

In this paper, a predator–prey reaction–diffusion system with one resource and two consumers is considered. Assume that one consumer species exhibits Holling II functional response while the other consumer species exhibits Beddington–DeAngelis functional response, and they compete for the common resource. First, it is proved that the unique positive constant steady state is stable for the ODE system and the reaction–diffusion system. Second, a prior estimates of positive steady state is given. Finally, the non-existence of non-constant positive steady state, the existence and bifurcation of non-constant positive steady state are studied.
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1. Introduction

In this paper, we are interested in a predator–prey system with one resource and two consumers. We assume that the first consumer species feeds upon the resource according to the Holling II functional response while the second consumer species feeds on the resource following the Beddington–DeAngelis functional response [1–3], and they compete for the common resource. The model is a system of three differential equations of the form

$$\begin{cases} \dfrac{du_1}{dt} = ru_1\left(1-\dfrac{u_1}{K}\right) - \dfrac{au_1u_2}{1+bu_1} - \dfrac{Au_1u_3}{1+Bu_1+Cu_3}, \\ \dfrac{du_2}{dt} = u_2\left(-m+\dfrac{eu_1}{1+bu_1}\right), \\ \dfrac{du_3}{dt} = u_3\left(-M+\dfrac{Eu_1}{1+Bu_1+Cu_3}\right), \end{cases} \tag{1.1}$$

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where $a, b, e, m, r, A, B, C, E, K, M$ are positive constants, $u_1(t), u_2(t), u_3(t)$ represent the density of one resource species and two consumer species at time t, respectively. By the following scaling:

$$\frac{u_1}{K} \mapsto u_1, \quad u_2 \mapsto u_2, \quad u_3 \mapsto u_3, \quad rt \mapsto t, \quad \frac{a}{r} \mapsto a, \quad bK \mapsto b, \quad \frac{A}{r} \mapsto A, \quad BK \mapsto B, \quad C \mapsto C,$$
$$\frac{m}{r} \mapsto m, \quad \frac{eK}{m} \mapsto e, \quad \frac{M}{r} \mapsto M, \quad \frac{EK}{M} \mapsto E,$$

system (1.1) takes the form

$$\begin{cases} \dfrac{du_1}{dt} = u_1(1-u_1) - \dfrac{au_1u_2}{1+bu_1} - \dfrac{Au_1u_3}{1+Bu_1+Cu_3}, \\ \dfrac{du_2}{dt} = mu_2\left(-1 + \dfrac{eu_1}{1+bu_1}\right), \\ \dfrac{du_3}{dt} = Mu_3\left(-1 + \dfrac{Eu_1}{1+Bu_1+Cu_3}\right). \end{cases} \tag{1.2}$$

It is known that the solutions of system (1.2) are non-negative and bounded for all $t \geqslant 0$ when the initial value condition $u_1(0) \geqslant 0, u_2(0) \geqslant 0, u_3(0) \geqslant 0$ holds [3], and it is obvious that problem (1.2) has a constant positive solution if and only if

$$e > b, \quad E - B > e - b, \quad CE(e-b-1) - A(e-b)\big[(E-B) - (e-b)\big] > 0. \tag{1.3}$$

Moreover, when (1.3) holds, the constant positive solution $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is uniquely given by

$$\begin{cases} \tilde{u}_1 = \dfrac{1}{e-b}, \\ \tilde{u}_2 = \left[(1-\tilde{u}_1) - \dfrac{A\tilde{u}_3}{E\tilde{u}_1}\right]\dfrac{e\tilde{u}_1}{a} = \dfrac{e\{CE(e-b-1) - A(e-b)[(E-B)-(e-b)]\}}{aEC(e-b)^2}, \\ \tilde{u}_3 = \dfrac{(E-B)\tilde{u}_1 - 1}{C} = \dfrac{(E-B)-(e-b)}{C(e-b)}. \end{cases} \tag{1.4}$$

This constant positive solution is also called positive steady-state solution or steady state of (1.2).

Now, if the resource and two consumer species are confined to a fixed bounded domain Ω in R^N with smooth boundary, and their densities are spatially inhomogeneous, from (1.2), we are led to consider the following reaction–diffusion system:

$$\begin{cases} u_{1t} - d_1\Delta u_1 = u_1(1-u_1) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1+Cu_3} \triangleq G_1(\mathbf{u}), & x \in \Omega,\ t > 0, \\ u_{2t} - d_2\Delta u_2 = mu_2(-1 + \frac{eu_1}{1+bu_1}) \triangleq G_2(\mathbf{u}), & x \in \Omega,\ t > 0, \\ u_{3t} - d_3\Delta u_3 = Mu_3(-1 + \frac{Eu_1}{1+Bu_1+Cu_3}) \triangleq G_3(\mathbf{u}), & x \in \Omega,\ t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) \geqslant 0, \quad i = 1,2,3, & x \in \Omega. \end{cases} \tag{1.5}$$

In the above, ν is the outward unit normal vector of the boundary $\partial\Omega$ and the homogeneous Neumann boundary condition is being considered. The constants d_i, $i = 1, 2, 3$ which are the diffusion coefficients, are positive, and the initial data $u_1(x,0), u_2(x,0), u_3(x,0)$ are continuous functions.

The system (1.5) arises in mathematical biology as a predator–prey model of three species which are interacting each other and migrating in the same habitat Ω. The corresponding ODE system (1.1) was proposed and studied in [3], where the explanations of the ecological background of this model can be found as well. The research of predator–prey models has a long histories. We can refer to [3–13] and references therein for a brief review of the development along this line. The major objective of this paper is to study the existence of non-constant positive steady-state solutions of (1.5). The Leray–Schauder degree theorem and bifurcation technique are our key tools to obtain the main results in this paper. In fact, the existence of positive steady-state solutions of reactive diffusion predator–prey system has been studied by the degree theorem and bifurcation technique in many works, see, for example, [10,13,16,21] for the homogeneous Dirichlet boundary condition, and [12,17,18,22,23] for homogeneous Neumann boundary conditions.

In the case that the consumer $u_2 \equiv 0$, or $u_3 \equiv 0$, system (1.5) reduces to a two species prey–predator model which has received extensive study. See, for example, while $u_2 \equiv 0$, in [18], the authors verified the dissipation, persistence, stability of non-negative constant steady state and the existence of non-constant positive steady state of the diffusion equation. In [19] a predator–prey dynamic model with the Beddington–DeAngelis functional response and Robin boundary condition has been studied.

In [20], Wonlyul Ko and Kimun Ryu studied a predator–prey model with Holling type II functional response incorporating a prey refuge under homogeneous Neumann boundary condition while $u_3 \equiv 0$. Moreover, they investigate the asymptotic behavior of spatially inhomogeneous solutions and the local existence of periodic solutions. Yihong Du and Yuan Lou have studied the similar system of (1.5) with the Dirichlet boundary conditions [21] and the Neumann boundary conditions [22] while $u_3 \equiv 0$. They discussed the non-existence and the existence of the positive steady-state solution.

This paper will be organized as follows. In Section 2, we will be proved that if the parameters A, B, C, E, e, b satisfy (1.3) and

$$E^2C\big[b(e-b-1)-e\big]+A(e-b)(Be-bE)\big[(E-B)-(e-b)\big]<0, \tag{1.6}$$

then the equilibrium solution $\tilde{\mathbf{u}}$ of (1.2) is locally asymptotically stable. In Section 3, we prove that if (1.3) and (1.6) hold then the constant positive steady state $\tilde{\mathbf{u}}$ of (1.5) is also locally asymptotically stable. The methods of Sections 2 and 3 to study local stability are based on local linearization techniques. In Section 4, a priori upper and lower bounds for positive steady states of (1.5) are established. In Sections 5–7, the non-existence of non-constant positive steady states, the existence and bifurcation of non-constant positive steady state of (1.5) are studied.

2. Stability of the positive steady states for the ODE system

In this section, we discuss the local stability of the positive steady state $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ for the ODE system (1.2). Throughout this paper, by saying that $\mathbf{u} = (u_1, u_2, u_3)$ is positive, we mean that $u_i > 0$, $i = 1, 2, 3$. Let $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ be a positive solution of (1.2). It is easy to see that $u_1(t), u_2(t)$ and $u_3(t)$ are bounded [3]. The main result of this section is the following.

Theorem 2.1. *If the parameters b, e, A, B, C and E satisfy* (1.3) *and* (1.6)*, then the steady-state solution $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$ is locally asymptotically stable.*

Proof. We shall use Routh–Hurwitz criterion [28] to prove our result. Let $\mathbf{G}(\mathbf{u}) = (G_1(\mathbf{u}), G_2(\mathbf{u}), G_3(\mathbf{u}))$, then problem (1.2) can be written as

$$\frac{d\mathbf{u}}{dt} = \mathbf{G}(\mathbf{u}).$$

Denote

$$\mathbf{G}_u(\tilde{\mathbf{u}}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

which stands for the derivative of $\mathbf{G}(\mathbf{u})$ at the steady state $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$. Since the parameters satisfy (1.3) and (1.6), through a direct computation, we can obtain that

$$\begin{cases} a_{11} = -\tilde{u}_1 + \dfrac{ab\tilde{u}_2}{e^2\tilde{u}_1} + \dfrac{AB\tilde{u}_3}{E^2\tilde{u}_1} = \dfrac{E^2C[b(e-b-1)-e]+A(e-b)(Be-bE)[(E-B)-(e-b)]}{eCE^2(e-b)} < 0, \\ a_{12} = -\dfrac{a\tilde{u}_1}{1+b\tilde{u}_1} = -\dfrac{a}{e} < 0, \quad a_{13} = -\dfrac{A\tilde{u}_3}{E\tilde{u}_1} < 0, \quad a_{21} = \dfrac{em\tilde{u}_2}{(1+b\tilde{u}_1)^2} = \dfrac{\tilde{u}_2}{me\tilde{u}_1} > 0, \quad a_{22} = a_{23} = 0, \\ a_{31} = \dfrac{ME\tilde{u}_3(1+C\tilde{u}_3)}{(1+B\tilde{u}_1+C\tilde{u}_3)^2} > 0, \quad a_{32} = 0, \quad a_{33} = -\dfrac{MEC\tilde{u}_1\tilde{u}_3}{(1+B\tilde{u}_1+C\tilde{u}_3)^2} < 0. \end{cases} \tag{2.1}$$

In fact, $a_{11} < 0$ is equivalent to (1.6) when $e > b$. The characteristic polynomial of $\mathbf{G}_u(\tilde{\mathbf{u}})$ can be written as

$$\varphi(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3.$$

From (2.1) one can calculate that

$$\begin{cases} A_1 = -(a_{11} + a_{33}) > 0, \\ A_2 = a_{11}a_{33} - a_{12}a_{21} - a_{31}a_{13} > 0, \\ A_3 = -\{\det \mathbf{G}_u(\tilde{\mathbf{u}})\} = a_{21}a_{12}a_{33} > 0. \end{cases} \tag{2.2}$$

Then, using (1.6) and (2.1), a direct calculation yields

$$\begin{aligned} A_1A_2 - A_3 &= -(a_{11} + a_{33})(a_{11}a_{33} - a_{12}a_{21} - a_{13}a_{31}) - a_{21}a_{12}a_{33} \\ &= -a_{11}^2a_{33} + a_{11}a_{12}a_{21} + a_{11}a_{31}a_{13} - a_{11}a_{33}^2 + a_{33}a_{31}a_{13} \\ &> 0. \end{aligned}$$

From the Routh–Hurwitz criterion, we can conclude that the characteristic polynomial of $\mathbf{G}_u(\tilde{\mathbf{u}})$ has only roots with negative real parts, and so $\tilde{\mathbf{u}}$ is local asymptotically stable. □

3. Stability of the constant positive steady states for the reaction–diffusion system

In this section, we discuss the local stability of the constant positive steady state $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ for the reaction–diffusion system (1.5). Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and $\mathbf{E}(\mu_i)$ be eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$. Let $\mathbf{X} = \{\mathbf{u} \in [C^1(\overline{\Omega})]^3 \mid \partial_\nu \mathbf{u} = 0 \text{ on } \partial\Omega\}$, $\{\phi_{ij};\ j = 1, \ldots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in R^3\}$. Then,

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}. \tag{3.1}$$

Theorem 3.1. *Assume that the parameters A, B, C, E, b, e satisfy* (1.3) *and* (1.6). *Then the constant positive steady state $\tilde{\mathbf{u}}$ of* (1.5) *is uniformly asymptotically stable.*

Proof. Let $\mathscr{D} = \mathrm{diag}(d_1, d_2, d_3)$, $L = \mathscr{D}\Delta + \mathbf{G}_u(\tilde{\mathbf{u}})$. The linearization of (1.5) at $\tilde{\mathbf{u}}$ is $\mathbf{u}_t = L\mathbf{u}$. For each $i \geqslant 0$, $\mathbf{X}_i$ is invariant under the operator L, and λ is an eigenvalue of L on $\mathbf{X}_i$ if and only if it is an eigenvalue of the matrix $-\mu_i\mathscr{D} + \mathbf{G}_u(\tilde{u})$ is given by

$$\psi_i(\lambda) = \lambda^3 + B_{1i}\lambda^2 + B_{2i}\lambda + B_{3i},$$

with

$$\begin{aligned} B_{1i} &= \mu_i(d_1 + d_2 + d_3) + A_1, \\ B_{2i} &= \mu_i^2(d_1d_2 + d_1d_3 + d_2d_3) - \mu_i\big[a_{33}d_1 + (a_{11} + a_{33})d_2 + a_{11}d_3\big] + A_2, \\ B_{3i} &= \mu_i^3 d_1d_2d_3 - \mu_i^2(d_1d_2a_{33} + d_2d_3a_{11}) + \mu_i\big[(a_{11}a_{33} - a_{13}a_{31})d_2 - a_{21}a_{12}d_3\big] + A_3, \end{aligned}$$

where a_{ij}, A_i are as given in (2.1), (2.2), respectively. In view of (2.1), it follows that $B_{1i}, B_{2i}, B_{3i} > 0$. Through a series of calculation, we have that

$$B_{1i}B_{2i} - B_{3i} = M_1\mu_i^3 + M_2\mu_i^2 + M_3\mu_i + A_1A_2 - A_3,$$

in which

$$\begin{aligned}
M_1 &= (d_1d_2 + d_1d_3 + d_2d_3)(d_1 + d_2 + d_3) - d_1d_2d_3 > 0,\\
M_2 &= -(a_{11} + a_{33})(d_1d_2 + d_2d_3 + d_1d_3) - \big[a_{33}d_1^2 + a_{33}(d_1d_2 + d_1d_3) + (a_{11} + a_{33})(d_1d_2 + d_2d_3)\\
&\quad + (a_{11} + a_{33})d_2 + a_{11}(d_3d_1 + d_3d_2) + a_{11}d_3^2\big] + d_1d_2a_{33} + d_2d_3a_{11}\\
&= -(a_{11} + a_{33})\big(2d_1d_2 + 2d_2d_3 + d_1d_3 + d_2^2\big) - a_{33}d_1(d_1 + d_3) - a_{11}d_3(d_1 + d_3) > 0,\\
M_3 &= (a_{11}a_{33} - a_{12}a_{21} - a_{31}a_{13})(d_1 + d_2 + d_3) + (a_{11} + a_{33})\big[a_{33}d_1 + (a_{11} + a_{33})d_2 + a_{11}d_3\big]\\
&\quad - \big[(a_{11}a_{33} - a_{13}a_{31})d_2 - a_{21}a_{12}d_3\big]\\
&= \big(2a_{11}a_{33} - a_{12}a_{21} + a_{33}^2a_{31}a_{13}\big)d_1 + \big(2a_{11}a_{33} - a_{12}a_{21} + a_{11}^2 + a_{33}^2\big)d_2\\
&\quad + (2a_{11}a_{33} - a_{31}a_{13} + a_{11}^2)d_3 > 0.
\end{aligned}$$

Recall that $A_1A_2 - A_3 > 0$, we conclude that $B_{1i}B_{2i} - B_{3i} > 0$ for all $i \geqslant 0$. It follows from the Routh–Hurwitz criterion that, for each $i \geqslant 0$, the three roots $\lambda_{1,i}, \lambda_{2,i}, \lambda_{3,i}$ of $\psi_i(\lambda) = 0$ all have negative real parts.

In the following we shall prove that there exists a positive constant δ such that

$$\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\} \leqslant -\delta, \quad \text{for all } i \geqslant 1. \tag{3.2}$$

Consequently, the spectrum of L, which consists of eigenvalues, lies in $\{\operatorname{Re}\lambda \leqslant -\delta\}$. The local stability of $\tilde{\mathbf{u}}$ then follows by applying Theorem 5.1.1 of [24, p. 98].

Now, we will prove (3.2). Let $\lambda = \mu_i\zeta$, then

$$\psi_i(\lambda) = \mu_i^3\zeta^3 + B_{1i}\mu_i^2\zeta^2 + B_{2i}\mu_i\zeta + B_{3i} \triangleq \tilde{\psi}_i(\zeta).$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty}\big\{\tilde{\psi}_i(\zeta)/\mu_i^3\big\} = \zeta^3 + (d_2 + d_2 + d_3)\zeta^2 + (d_1d_2 + d_1d_3 + d_2d_3)\zeta + d_1d_2d_3 \triangleq \bar{\psi}(\zeta).$$

Applying the Routh–Hurwitz criterion, it follows that the three roots $\zeta_1, \zeta_2, \zeta_3$ of $\bar{\psi}(\zeta) = 0$ all have negative real parts. Thus, there exists a positive constant $\bar{\delta}$ such that $\operatorname{Re}\{\zeta_1\}, \operatorname{Re}\{\zeta_2\}, \operatorname{Re}\{\zeta_3\} \leqslant -\bar{\delta}$. By continuity, we see that there exists i_0 such that the three roots $\zeta_{i1}, \zeta_{i2}, \zeta_{i3}$ of $\bar{\psi}_i(\zeta) = 0$ satisfy

$$\operatorname{Re}\{\zeta_{i,1}\}, \operatorname{Re}\{\zeta_{i,2}\}, \operatorname{Re}\{\zeta_{i,3}\} \leqslant -\bar{\delta}/2, \quad \text{for all } i \geqslant i_0.$$

In turn, $\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\} \leqslant -\mu_i\bar{\delta}/2 \leqslant -\bar{\delta}/2$, for all $i \geqslant i_0$.

Let

$$-\tilde{\delta} = \max_{1\leqslant i\leqslant i_0}\big\{\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\}\big\},$$

then $\tilde{\delta} > 0$, and (3.2) holds for

$$\delta = \min\{\tilde{\delta}, \bar{\delta}/2\}.$$

The proof is complete. □

As a consequence of Theorem 3.1, problem (1.5) has no non-constant positive steady state in some neighborhood of $\tilde{\mathbf{u}}$ if (1.3) and (1.6) hold.

4. A priori estimates of positive steady state

The corresponding steady-state problem of (1.5) is

$$\begin{cases}
-d_1\Delta u_1 = G_1(\mathbf{u}), & x \in \Omega,\\
-d_2\Delta u_2 = G_2(\mathbf{u}), & x \in \Omega,\\
-d_3\Delta u_3 = G_3(\mathbf{u}), & x \in \Omega,\\
\partial_\nu u_1 = \partial_\nu u_2 = \partial_\nu u_3 = 0, & x \in \partial\Omega.
\end{cases} \tag{4.1}$$

In the sequel, the generic constants $C_1, C_2, C_*, \underline{C}, \overline{C}$, etc., will depend on the domain Ω and the dimension N. However, as Ω and the dimension N are fixed, we will not mention the dependence explicitly. Also, for convenience, we shall write Λ instead of the collective constants $(a, b, e, m, A, B, C, E, M)$.

The main purpose of this section is to give a priori positive upper and lower bounds for the positive solutions of (4.1). To this aim, we will cite two known results. The first is due to Lin, Ni and Takagi [25], and the second to Lou and Ni [26].

Proposition 4.1 *(Harnack inequality). (See [25].) Let $\omega \in C^2(\Omega) \cap C^1(\Omega)$ be a classical positive solution to $\Delta\omega(x) + c(x)\omega(x) = 0$ in Ω, where $c(x) \in C(\overline{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant $C_* = C_*(N, \Omega, \|c\|_\infty)$ such that*

$$\max_{\overline{\Omega}} \omega \leqslant C_* \min_{\overline{\Omega}} \omega$$

Proposition 4.2 *(Maximum principle). (See [26].) Suppose that $g \in C(\Omega \times \mathbf{R}^1)$ and $b_j \in C(\overline{\Omega})$, $j = 1, 2, \ldots, N$.*

(i) *If $\omega \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta\omega(x) + \sum_{j=1}^{N} b_j(x)\omega_{x_j} + g\big(x, \omega(x)\big) \geqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu \omega \leqslant 0 \quad \text{on } \partial\Omega,$$

and $\omega(x_0) = \max_{\overline{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \geqslant 0$.

(ii) *If $\omega \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta\omega(x) + \sum_{j=1}^{N} b_j(x)\omega_{x_j} + g\big(x, \omega(x)\big) \leqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu \omega \geqslant 0 \quad \text{on } \partial\Omega,$$

and $\omega(x_0) = \min_{\overline{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \leqslant 0$.

In this paper, by classical solutions, we mean solutions in $C^2(\Omega) \cap C^1(\overline{\Omega})$. The result of upper bounds can be stated as follows:

Theorem 4.1 *(Upper bounds). For any positive classical solution (u_1, u_2, u_3) of (4.1),*

$$\max_{\overline{\Omega}} u_1 \leqslant 1, \qquad \max_{\overline{\Omega}} u_2 \leqslant \frac{e(4md_1 + d_2)}{4ad_2}, \qquad \max_{\overline{\Omega}} u_3 \leqslant \frac{E - B - 1}{C}. \tag{4.2}$$

Proof. Since

$$u_1(1 - u_1) - \frac{au_1u_2}{1 + bu_1} - \frac{Au_1u_3}{1 + Bu_1 + Cu_3} \leqslant u_1(1 - u_1),$$

the first result follows easily from the maximum principle. Then we can conclude the third inequality of (4.2) by a direct application.

Let $\omega = med_1u_1 + ad_2u_2$, then we can conclude that

$$\begin{cases} -\Delta\omega = meu_1(1 - u_1) - \frac{Ameu_1u_3}{1+Bu_1+Cu_3} - mau_2, & x \in \Omega, \\ \partial_\nu\omega = 0, & x \in \partial\Omega. \end{cases}$$

Let $\omega(x_0) = \max_{\overline{\Omega}} \omega(x)$. By the application of the maximum principle, it yields

$$mau_2(x_0) \leqslant meu_1(1 - u_1) - \frac{Ameu_1u_3}{1 + Bu_1u_3} \leqslant \frac{me}{4}.$$

Consequently,

$$ad_2 \max_{\overline{\Omega}} u_2(x) \leqslant \max_{\overline{\Omega}} \omega(x) = \omega(x_0) = med_1u_1(x_0) + ad_2u_2(x_0) \leqslant med_1 + \frac{ed_2}{4},$$

and hence

$$\max_{\overline{\Omega}} u_2(x) \leqslant \frac{e(4md_1+d_2)}{4ad_2}.$$

The proof is completed. □

Theorem 4.2 *(Lower bounds). Let Λ and $\underline{d}_1, \underline{d}_2, \underline{d}_3$ be fixed positive constants. Assume that $(d_1, d_2, d_3) \in [\underline{d}_1, \infty) \times [\underline{d}_2, \infty) \times [\underline{d}_3, \infty)$, and*

$$\min\left\{\frac{4d_2 - 4med_1 - ed_2}{4d_2}, \frac{C(E-B) - A(E-B-1)}{C(E-B)}\right\} > \frac{1}{e-b}. \tag{4.3}$$

Then there exists a positive constant $\underline{C} = \underline{C}(\Lambda, \underline{d}_1, \underline{d}_2, \underline{d}_3)$, such that every positive classical solution (u_1, u_2, u_3) of (4.1) *satisfies*

$$\min_{\overline{\Omega}} u_i(x) > \underline{C}, \quad i = 1, 2, 3. \tag{4.4}$$

Proof. Let

$$c_1(x) = d_1^{-1}\left(1 - u_1 - \frac{au_2}{1+bu_1} - \frac{Au_3}{1+Bu_1+Cu_3}\right),$$
$$c_2(x) = d_2^{-1} m\left(-1 + \frac{eu_1}{1+bu_1}\right),$$
$$c_3(x) = d_3^{-1} M\left(-1 + \frac{Eu_1}{1+Bu_1+Cu_3}\right).$$

Then, in view of (4.2), there exists a positive constant $\overline{C}(d, \Lambda)$ such that $\|c_1\|_\infty, \|c_2\|_\infty, \|c_3\|_\infty \leqslant \overline{C}$, if $d_1, d_2, d_3 \geqslant d$. Thus, as u_1, u_2, u_3 satisfy

$$\Delta u_i + c_i(x)u_i = 0, \quad x \in \Omega; \qquad \frac{\partial u_i}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad i = 1, 2, 3.$$

The Harnack inequality in Proposition 4.1 shows that there exists a positive constant $C_* = C_*(\Lambda, d)$ such that

$$\max_{\overline{\Omega}} u_i \leqslant C_* \min_{\overline{\Omega}} u_i, \quad i = 1, 2, 3. \tag{4.5}$$

Now, suppose, on the contrary, that (4.4) does not hold. Then there exists a sequence $\{d_{1i}, d_{2i}, d_{3i}\}_{i=1}^\infty$ with $d_{1i}, d_{2i}, d_{3i} \in [\underline{d}_1, \infty) \times [\underline{d}_2, \infty) \times [\underline{d}_3, \infty)$ such that the corresponding positive solutions (u_{1i}, u_{2i}, u_{3i}) of (4.1) satisfy

$$\max_{\overline{\Omega}} u_{1i} \to 0 \quad \text{or} \quad \max_{\overline{\Omega}} u_{2i} \to 0 \quad \text{or} \quad \max_{\overline{\Omega}} u_{3i} \to 0, \quad \text{as } i \to \infty. \tag{4.6}$$

By the maximum principle, $u_{1i} \leqslant 1$. Integrating by parts, we obtain that

$$\begin{cases} \displaystyle\int_\Omega \left[u_{1i}(1-u_{1i}) - \frac{au_{1i}u_{2i}}{1+bu_{1i}} - \frac{Au_{1i}u_{3i}}{1+Bu_{1i}+Cu_{3i}}\right] dx = 0, \\ \displaystyle\int_\Omega mu_{2i}\left(-1 + \frac{eu_{1i}}{1+bu_{1i}}\right) dx = 0, \\ \displaystyle\int_\Omega Mu_{3i}\left(-1 + \frac{Eu_{1i}}{1+Bu_{1i}+Cu_{3i}}\right) dx = 0, \end{cases} \tag{4.7}$$

for $i = 1, 2, \ldots$. The standard regularity theorem for the elliptic equations yields that there exists a subsequence of $\{u_{1i}, u_{2i}, u_{3i}\}_{i=1}^\infty$, which we shall still denote by $\{u_{1i}, u_{2i}, u_{3i}\}_{i=1}^\infty$, and three non-negative functions $u_1, u_2, u_3 \in C^2(\overline{\Omega})$, such that $(u_{1i}, u_{2i}, u_{3i}) \to (u_1, u_2, u_3)$ in $[C^2(\overline{\Omega})]^3$ as $i \to \infty$. By (4.6), we note that $u_1 \equiv 0$ or $u_2 \equiv 0$

or $u_3 \equiv 0$. Moreover, we assume that $(d_{1i}, d_{2i}, d_{3i}) \to (\bar{d}_1, \bar{d}_2, \bar{d}_3) \in [\underline{d}_1, \infty) \times [\underline{d}_2, \infty) \times [\underline{d}_3, \infty)$, and $\bar{d}_1, \bar{d}_2$ satisfy (4.3), i.e.

$$\frac{4\bar{d}_2 - 4me\bar{d}_1 - e\bar{d}_2}{4\bar{d}_2} > \frac{1}{e-b}. \tag{4.8}$$

Let $i \to \infty$ in (4.7) we obtain that

$$\begin{cases} \int_\Omega \left[u_1(1-u_1) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1+Cu_3}\right] dx = 0, \\ \int_\Omega mu_2\left(-1+\frac{eu_1}{1+bu_1}\right) dx = 0, \\ \int_\Omega Mu_3\left(-1+\frac{Eu_1}{1+Bu_1+Cu_3}\right) dx = 0. \end{cases}$$

We now consider the following three cases.

Case 1. $u_1 \equiv 0$.

Since $u_{1i} \to u_1$, as $i \to \infty$. Then

$$-1 + \frac{eu_{1i}}{1+bu_{1i}} < 0 \quad \text{on } \overline{\Omega}, \text{ for all } i \gg 1.$$

Integrating the differential equation for u_{2i} over Ω by parts, we have

$$0 = d_{2i} \int_{\partial\Omega} \partial_\nu u_{2i}\, ds = -d_{2i} \int_\Omega \Delta u_{2i}\, dx = \int_\Omega u_{2i}\left(-1+\frac{eu_{1i}}{1+bu_i}\right) dx < 0, \quad \text{for all } i \gg 1,$$

which is a contradiction.

Case 2. $u_2 \equiv 0$, $u_1 \not\equiv 0$ on $\overline{\Omega}$, then the Hopf boundary lemma gives $u_1 > 0$ on $\overline{\Omega}$.

In this case, u_1 and u_3 satisfy

$$-\bar{d}_1 \Delta u_1 = u_1\left[(1-u_1) - \frac{Au_3}{1+Bu_1+Cu_3}\right], \quad x \in \Omega; \qquad \partial_\nu u_1 = 0, \quad x \in \partial\Omega. \tag{4.9}$$

Applying Proposition 4.2 and the third inequality of (4.2), let $u_1(x_0) = \min_{\overline{\Omega}} u_1(x)$, it follows from (4.9) that

$$1 - u_1(x_0) - \frac{Au_3(x_0)}{1+Bu_1(x_0)+Cu_3(x_0)} \leqslant 0,$$

i.e.

$$u_1(x_0) \geqslant 1 - \frac{Au_3(x_0)}{1+Bu_1(x_0)+Cu_3(x_0)} \geqslant 1 - \frac{Au_3(x_0)}{1+Cu_3(x_0)} \geqslant 1 - \frac{A \cdot \frac{E-B-1}{C}}{(E-B)},$$
$$u_1(x_0) \geqslant \frac{C(E-B) - A(E-B-1)}{C(E-B)}.$$

Using the given assumptions $\frac{C(E-B)-A(E-B-1)}{C(E-B)} > \frac{1}{e-b}$, it is easy to see that

$$-1 + \frac{eu_{1i}}{1+bu_{1i}} > 0 \quad \text{on } \overline{\Omega}, \text{ for all } i \gg 1.$$

Integrating the differential equation for u_{2i} over Ω by parts, we have

$$0 = d_{2i} \int_{\partial\Omega} \partial_\nu u_{2i}\, ds = -d_{2i} \int_\Omega \Delta u_{2i}\, dx = \int_\Omega u_{2i}\left(-1+\frac{eu_{1i}}{1+bu_i}\right) dx > 0, \quad \text{for all } i \gg 1,$$

which is a contradiction.

Case 3. $u_3 \equiv 0$, $u_1 \neq 0$, $u_2 \neq 0$, on $\overline{\Omega}$, then the Hopf boundary lemma gives $u_1 > 0$, $u_2 > 0$ on $\overline{\Omega}$, and u_1, u_2 satisfy

$$-\bar{d}_1 \Delta u_1 = u_1\left[(1-u_1) - \frac{au_2}{1+bu_1}\right], \quad x \in \Omega, \qquad \partial_\nu u_1 = 0, \quad x \in \partial\Omega. \tag{4.10}$$

Applying Proposition 4.2 and the second inequality of (4.2), let $u_1(x_0) = \min_{\overline{\Omega}} u_1(x)$, it follows from (4.10) that

$$u_1(x_0) \geqslant 1 - \frac{au_2(x_0)}{1+bu_1(x_0)} \geqslant 1 - \frac{4me\bar{d}_1 + e\bar{d}_2}{4\bar{d}_2},$$
$$u_1(x_0) \geqslant \frac{4\bar{d}_2 - 4me\bar{d}_1 - e\bar{d}_2}{4\bar{d}_2}.$$

Since $\bar{d}_1, \bar{d}_2$ satisfy (4.8), it is easy to see that

$$-1 + \frac{eu_{1i}}{1+bu_{1i}} > 0 \quad \text{on } \overline{\Omega}, \text{ for all } i \gg 1.$$

Integrating the differential equation for u_{2i} over Ω by parts, we have

$$0 = d_{2i}\int_{\partial\Omega} \partial_\nu u_{2i}\,ds = -d_{2i}\int_\Omega \Delta u_{2i}\,dx = \int_\Omega u_{2i}\left(-1 + \frac{eu_{1i}}{1+bu_i}\right)dx > 0, \quad \text{for all } i \gg 1,$$

which is a contradiction. The proof of Theorem 4.2 is complete. □

5. Non-existence of non-constant positive steady state

In this section we shall discuss the non-constant positive solutions to problem (4.1) when the diffusion coefficient d_1 varies while the other parameters d_2, d_3, Λ are fixed.

Theorem 5.1. *Let d_2^* and d_3^* be fixed positive constants and satisfy $d_2^*\mu_1 \geqslant \frac{m(e-b-1)}{1+b}$, $d_3^*\mu_1 \geqslant \frac{M(E-B-1)}{1+B}$. Then there exists a positive constant $D_1 = D_1(\Lambda, d_2^*, d_3^*)$, such that, when $d_1 > D_1$, $d_2 \geqslant d_2^*$, and $d_3 \geqslant d_3^*$, problem* (4.1) *has no non-constant positive solution.*

Proof. For any $\varphi \in L^1(\Omega)$, let

$$\bar{\varphi} = 1/|\Omega| \int_\Omega \varphi\,dx. \tag{5.1}$$

Multiplying the differential equation (4.1) by $\mathbf{u} - \bar{\mathbf{u}}$, and then integrating over Ω by parts, we have

$$\begin{aligned}
\sum_{i=1}^3 \int_\Omega d_i|\nabla u_i|^2\,dx &= \sum_{i=1}^3 \int_\Omega \big(\mathbf{G}_i(u) - \mathbf{G}_i(\bar{u})\big)(u_i - \bar{u}_i) \\
&= \int_\Omega \Bigg\{(u_1-\bar{u}_1)^2\big[1-(u_1+\bar{u}_1)\big] \\
&\quad - \frac{au_2(u_1-\bar{u}_1)^2 + (a\bar{u}_1 + abu_1\bar{u}_1)(u_1-\bar{u}_1)(u_2-\bar{u}_2)}{(1+bu_1)(1+b\bar{u}_1)} \\
&\quad - \frac{(Au_1 + ABu_1\bar{u}_1)(u_3-\bar{u}_3)(u_1-\bar{u}_1) + (Au_3 + ACu_3\bar{u}_3)(u_1-\bar{u}_1)^2}{(1+Bu_1+Cu_3)(1+B\bar{u}_1+C\bar{u}_3)} \\
&\quad - m(u_2-\bar{u}_2)^2 + m\frac{eu_2(u_1-\bar{u}_1)(u_2-\bar{u}_2) + (ebu_1\bar{u}_1 + e\bar{u}_1)(u_2-\bar{u}_2)^2}{(1+bu_1)(1+b\bar{u}_1)} - M(u_3-\bar{u}_3)^2 \\
&\quad + \frac{(Eu_1 + EBu_1\bar{u}_1)(u_3-\bar{u}_3)^2 + (Eu_3 + ECu_3\bar{u}_3)(u_1-\bar{u}_1)(u_3-\bar{u}_3)}{(1+Bu_1+Cu_3)(1+B\bar{u}_1+C\bar{u}_3)}\Bigg\}\,dx
\end{aligned}$$

$$\leqslant \int_\Omega \left\{(1+C_1+C_2)(u_1-\bar{u}_1)^2 + \left(\frac{m(e-b-1)}{1+b}+\varepsilon_1\right)(u_2-\bar{u}_2)^2 + \left(\frac{M(E-B-1)}{1+B}+\varepsilon_2\right)(u_3-\bar{u}_3)^2\right\} dx, \tag{5.2}$$

for some positive constants $C_1 = C_1(\Lambda, d_2^*, d_3^*, \varepsilon_1)$, $C_2 = C_2(\Lambda, d_2^*, d_3^*, \varepsilon_2)$, where $\varepsilon_1, \varepsilon_2$ are the arbitrary small positive constants arising from Young's inequality.

In view of the Poincare inequality [29],

$$\mu_1 \int_\Omega (f-\bar{f})^2\,dx \leqslant \int_\Omega |\nabla f|^2\,dx,$$

where $\bar{f}$ is similar to (5.1), it follows from (5.2) that

$$\mu_1 \sum_{i=1}^{3} \int_\Omega d_i(u_i-\bar{u}_i)^2\,dx \leqslant \int_\Omega \left\{(1+C_1+C_2)(u_1-\bar{u}_1)^2 + \left(\frac{m(e-b-1)}{1+b}+\varepsilon_1\right)(u_2-\bar{u}_2)^2 + \left(\frac{M(E-B-1)}{1+B}+\varepsilon_2\right)(u_3-\bar{u}_3)^2\right\} dx. \tag{5.3}$$

Choose $\varepsilon_1, \varepsilon_2 > 0$ very small such that

$$\mu_1 d_2^* \geqslant \frac{m(e-b-1)}{1+b}+\varepsilon_1, \qquad \mu_1 d_3^* \geqslant \frac{M(E-B-1)}{1+B}+\varepsilon_2.$$

Then (5.3) implies that $u_2 = \bar{u}_2 =$ constant, $u_3 = \bar{u}_3 =$ constant, and $u_1 = \bar{u}_1 =$ constant if $d_1 > D_1 \triangleq \mu_1^{-1}(1+C_1+C_2)$. The proof is complete. □

6. Existence of non-constant positive steady states

In this section we discuss the existence of non-constant positive classical solutions to (4.1) when the diffusion coefficients d_2 vary while the parameters Λ, d_1, d_3 are kept fixed. Theorem 3.1 implies that when (1.3) holds and $a_{11} < 0$, then (4.1) has no non-constant positive classical solutions. In view of this reason, we shall restrict this discussion to the case where Λ satisfy (1.3) and $a_{11} > 0$.

First, we shall study the linearization of (4.1) at $\tilde{\mathbf{u}}$. Let $\mathbf{X}$ be as in Section 3, and define

$$\mathbf{X}^+ = \{\mathbf{u} \in \mathbf{X} \mid u_i > 0 \text{ on } \overline{\Omega},\ i = 1,2,3\},$$
$$\mathcal{B}(c) = \left\{\mathbf{u} \in \mathbf{X} \,\middle|\, c^{-1} < u_i < c \text{ on } \overline{\Omega},\ i = 1,2,3\right\},$$

where c is a positive constant that is guaranteed to exist by Theorems 4.1 and 4.2. Then (4.1) can be written as

$$\begin{cases} -\mathscr{D}\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

and $\mathbf{u}$ is a positive solution to (6.1) if and only if

$$\mathbf{F}(\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I}-\Delta)^{-1}\left\{\mathscr{D}^{-1}\mathbf{G}(\mathbf{u}) + \mathbf{u}\right\} = 0 \quad \text{in } \mathbf{X}^+,$$

where $(\mathbf{I}-\Delta)^{-1}$ is the inverse of $\mathbf{I}-\Delta$ in $\mathbf{X}$ with the homogeneous Neumann boundary condition. As $\mathbf{F}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathcal{B} = \mathcal{B}(c)$. The Leray–Schauder degree $\deg(\mathbf{F}(\cdot), 0, \mathcal{B})$ is well defined if $\mathbf{F}(\mathbf{u}) \neq 0$ on $\partial\mathcal{B}$.

Further, we note that

$$\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\left\{\mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\right\},$$

and recall that if $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$ is invertible, the index of $\mathbf{F}$ at $\tilde{\mathbf{u}}$ is defined as $\mathrm{index}(\mathbf{F}(\cdot), \tilde{\mathbf{u}}) = (-1)^\gamma$, where γ is the total number of eigenvalues with negative real parts (counting multiplicities) of $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$ [27].

We refer to the decomposition (3.1) in our following discussion of the eigenvalues of $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$. First, we note that, for each integer $i \geqslant 0$ and each integer $1 \leqslant j \leqslant \dim E(\mu_i)$, $\mathbf{X}_{ij}$ is invariant under $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$, and λ is an eigenvalue of $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$ on $\mathbf{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$\mathbf{I} - \frac{1}{1+\mu_i}\big[\mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\big] = \frac{1}{1+\mu_i}\big[\mu_i\mathbf{I} - \mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}})\big].$$

Thus, $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$ is invertible if and only if, for all $i \geqslant 0$, the matrix $\mathbf{I} - [1/(1+\mu_i)][\mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}]$ is non-singular.

Write

$$H(\mu) = H(\tilde{\mathbf{u}};\mu) \triangleq \det\big\{\mu\mathbf{I} - \mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}})\big\} = \frac{1}{d_1d_2d_3}\det\big\{\mu\mathscr{D} - \mathbf{G}_u(\tilde{\mathbf{u}})\big\}. \tag{6.2}$$

We note, furthermore, that if $H(\mu_i) \neq 0$, then for each $1 \leqslant j \leqslant \dim E(\mu_i)$, the number of negative eigenvalues of $\mathbf{D}_u\mathbf{F}(\tilde{\mathbf{u}})$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\mu_i) < 0$. From this, we can conclude the following result, it also can be found in [12,23].

Proposition 6.1. *Suppose that, for all $i \geqslant 0$, the matrix $\mu_i\mathbf{I} - \mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}})$ is non-singular. Then*

$$\operatorname{index}\big(\mathbf{F}(\cdot),\tilde{\mathbf{u}}\big) = (-1)^{\sigma}, \quad \textit{where } \sigma = \sum_{i\geqslant 0,\, H(\mu_i)<0} \dim E(\mu_i).$$

This proposition shows that σ and γ have the same parity in here. To calculate the index of $(\mathbf{F}(\cdot),\tilde{\mathbf{u}})$, we will consider carefully the sign of $H(\mu_i)$. The direct calculation gives

$$\det\big\{\mu\mathscr{D} - \mathbf{G}_u(\tilde{\mathbf{u}})\big\} = A_3(d_2)\mu^3 + A_2(d_2)\mu^2 + A_1(d_2)\mu - \det\big\{\mathbf{G}_u(\tilde{\mathbf{u}})\big\} \triangleq \mathcal{A}(d_2;\mu), \tag{6.3}$$

with

$$\begin{cases} A_3(d_2) = d_1d_2d_3, \quad A_2(d_2) = -(a_{33}d_1d_2 + a_{11}d_2d_3), \\ A_1(d_2) = a_{11}a_{33}d_2 - a_{31}a_{13}d_2 - a_{12}a_{21}d_3, \end{cases}$$

where a_{ij} are as given in (2.1).

We consider the dependence of $\mathcal{A}$ on d_2. Let $\tilde{\mu}_1(d_2)$, $\tilde{\mu}_2(d_2)$ and $\tilde{\mu}_3(d_2)$ be the three roots of $\mathcal{A}(d_2;\mu) = 0$. Then

$$\tilde{\mu}_1(d_2)\tilde{\mu}_2(d_2)\tilde{\mu}_3(d_2) = \det\big\{\mathbf{G}_u(\tilde{\mathbf{u}})\big\}.$$

The direct computation gives $\det\{\mathbf{G}_u(\tilde{\mathbf{u}})\} < 0$. Note that $A_3(d_2) > 0$. Thus, one of $\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)$ is real and negative, and the product of the other two is positive.

Consider the following limits:

$$\lim_{d_2\to\infty}\frac{A_3(d_2)}{d_2} = d_1d_3, \qquad \lim_{d_2\to\infty}\frac{A_2(d_2)}{d_2} = -a_{33}d_1 - a_{11}d_3, \qquad \lim_{d_2\to\infty}\frac{A_1(d_2)}{d_2} = a_{11}a_{33} - a_{31}a_{13},$$

$$\begin{aligned}\lim_{d_2\to\infty}\frac{\mathcal{A}(d_2)}{d_2} &= d_1d_3\mu^3 - (a_{33}d_1 + a_{11}d_3)\mu^2 + (a_{11}a_{33} - a_{31}a_{13})\mu \\ &= \mu\big[d_1d_3\mu^2 - (a_{33}d_1 + a_{11}d_3)\mu + a_{11}a_{33} - a_{31}a_{13}\big].\end{aligned}$$

If the parameters Λ, d_1, d_3 satisfy $a_{11}d_3 + a_{33}d_1 > 0$, we can establish the following:

Proposition 6.2. *Assume that* (1.3) *holds, and $a_{11} > 0$. Then there exists a positive constant D_2, such that when $d_2 \geqslant D_2$, the three roots $\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)$ of $\mathcal{A}(d_2;\mu) = 0$ are all real and satisfy*

$$\begin{aligned}&\lim_{d_2\to\infty}\tilde{\mu}_1(d_2) = \frac{a_{11}d_3 + a_{33}d_1 - \sqrt{(a_{11}d_3 - a_{33}d_1)^2 + 4a_{31}a_{13}d_1d_3}}{2d_1d_3} \triangleq \hat{\mu}, \\ &\lim_{d_2\to\infty}\tilde{\mu}_2(d_2) = 0, \\ &\lim_{d_2\to\infty}\tilde{\mu}_3(d_2) = \frac{a_{11}d_3 + a_{33}d_1 + \sqrt{(a_{11}d_3 - a_{33}d_1)^2 + 4a_{31}a_{13}d_1d_3}}{2d_1d_3} \triangleq \check{\mu} > 0.\end{aligned} \tag{6.4}$$

Moreover, if $a_{11}a_{33} - a_{31}a_{13} < 0$, *then*

$$\begin{cases} -\infty < \tilde{\mu}_1(d_2) < 0 < \tilde{\mu}_2(d_2) < \tilde{\mu}_3(d_2), \\ \mathcal{A}(d_2;\mu) < 0, \quad \text{when } \mu \in \big(-\infty, \tilde{\mu}_1(d_2)\big) \cup \big(\tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)\big), \\ \mathcal{A}(d_2;\mu) > 0, \quad \text{when } \mu \in \big(\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2)\big) \cup \big(\tilde{\mu}_3(d_2), +\infty\big). \end{cases} \tag{6.5a}$$

If $a_{11}a_{33} - a_{31}a_{13} > 0$, *then*

$$\begin{cases} -\infty < \tilde{\mu}_2(d_2) < 0 < \tilde{\mu}_1(d_2) < \tilde{\mu}_3(d_2), \\ \mathcal{A}(d_2;\mu) < 0, \quad \text{when } \mu \in \big(-\infty, \tilde{\mu}_2(d_2)\big) \cup \big(\tilde{\mu}_1(d_2), \tilde{\mu}_3(d_2)\big), \\ \mathcal{A}(d_2;\mu) > 0, \quad \text{when } \mu \in \big(\tilde{\mu}_2(d_2), \tilde{\mu}_1(d_2)\big) \cup \big(\tilde{\mu}_3(d_2), +\infty\big). \end{cases} \tag{6.5b}$$

Now we prove the existence of non-constant positive solutions of (4.1) for some Λ, $d_i, i = 1, 3$, when d_2 is sufficiently large.

Theorem 6.1. *Assume that the parameters* Λ, d_i, $i = 1, 3$, *are fixed,* $a_{11} > 0$, (1.3), (4.3) *hold, and one of the following conditions is satisfied*:

(i) $a_{11}a_{33} - a_{31}a_{13} < 0$, $\check{\mu} \in (\mu_n, \mu_{n+1})$ *for some* $n \geqslant 1$, *and the sum* $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)$ *is odd.*
(ii) $a_{11}a_{33} - a_{31}a_{13} > 0$, $\hat{\mu} \in (\mu_k, \mu_{k+1})$, $\check{\mu} \in (\mu_n, \mu_{n+1})$ *for some* $n \geqslant k \geqslant 1$, *and the sum* $\sigma_n = \sum_{i=k+1}^{n} \dim E(\mu_i)$ *is odd.*

Then there exists a positive constant D_2 *such that, if* $d_2 \geqslant D_2$, (4.1) *has at least one non-constant positive solution.*

Proof. If $a_{11}a_{33} - a_{31}a_{13} < 0$, by Proposition 6.2, there exists a positive constant D_2, such that when $d_2 \geqslant D_2$, (6.5a) holds and

$$0 = \mu_0 < \tilde{\mu}_2(d_2) < \mu_1, \qquad \tilde{\mu}_3(d_2) \in (\mu_n, \mu_{n+1}). \tag{6.6a}$$

We shall prove that for any $d_2 \geqslant D_2$, (4.1) has at least one non-constant positive solution. The proof, which is accomplished by contradiction, is based on the homotopy invariance of the topological degree. Suppose on the contrary that the assertion is not true for some $d_2 = \tilde{d}_2 \geqslant D_2$. In the sequel we fixed $d_2 = \tilde{d}_2$,

$$d_2^* = \frac{me}{\mu_1(1+b)}, \qquad d_3^* = \frac{ME}{\mu_1(1+B)}.$$

By Theorem 5.1, we obtain a positive constant $D_1 = D_1(\Lambda, d_2^*, d_3^*)$. Fix $\hat{d}_2 \geqslant d_2^*$, $\hat{d}_3 \geqslant \max\{d_3^*, d_3\}$, $\hat{d}_1 \geqslant \max\{D_1, d_1\}$. For $t \in [0, 1]$, define $\mathscr{D}(t) = \operatorname{diag}(d_1(t), d_2(t), d_3(t))$ with $d_i(t) = td_i + (1-t)\hat{d}_i$, $i = 1, 2, 3$, and consider the problem

$$\begin{cases} -\mathscr{D}(t)\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{6.7}$$

Then $\mathbf{u}$ is a non-constant positive solution of (4.1) if and only if it is a positive solution of (6.7) for $t = 1$. It is obvious that $\tilde{\mathbf{u}}$ is the unique constant positive solution of (6.7) for any $0 \leqslant t \leqslant 1$. For any $0 \leqslant t \leqslant 1$, $\mathbf{u}$ is a positive solution of (6.7) if and only if

$$\mathbf{F}(t;\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\big\{\mathscr{D}^{-1}(t)\mathbf{G}(\mathbf{u}) + \mathbf{u}\big\} = 0 \quad \text{in } \mathbf{X}^+.$$

It is obvious that $\mathbf{F}(1;\mathbf{u}) = \mathbf{F}(\mathbf{u})$, Theorem 5.1 shows that $\mathbf{F}(0;\mathbf{u}) = 0$ has only the positive solution $\tilde{\mathbf{u}}$ in $\mathbf{X}^+$. By a direct computation,

$$\mathbf{D}_u\mathbf{F}(t;\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\big\{\mathscr{D}^{-1}(t)\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\big\}.$$

In particular,

$$\mathbf{D}_u\mathbf{F}(0;\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\big\{\widehat{\mathscr{D}}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\big\},$$
$$\mathbf{D}_u\mathbf{F}(1;\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\big\{\mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\big\} = \mathbf{D}_u F(\tilde{\mathbf{u}}),$$

where $\widehat{\mathscr{D}} = \mathrm{diag}(\hat{d}_1, \hat{d}_2, \hat{d}_3)$. From (6.2) and (6.3) we see that

$$H(\mu) = \frac{1}{d_1 d_2 d_3}\mathcal{A}(d_2;\mu). \tag{6.8}$$

In view of (6.5a) and (6.6a), it follows from (6.8) that

$$\begin{cases} H(\mu_0) = H(0) > 0, \\ H(\mu_i) < 0, \quad 1 \leqslant i \leqslant n, \\ H(\mu_{i+1}) > 0, \quad i \geqslant n+1. \end{cases}$$

Therefore, 0 is not an eigenvalue of the matrix $\mu_i\mathbf{I} - \mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}})$ for all $i \geqslant 0$, and

$$\sum_{i\geqslant 0, H(\mu_i)<0} \dim E(\mu_i) = \sum_{i=1}^{n} \dim E(\mu_i) = \sigma_n, \quad \text{which is odd.}$$

Thanks to Proposition 6.1, we have

$$\mathrm{index}\big(\mathbf{F}(1;\cdot),\tilde{\mathbf{u}}\big) = (-1)^{\gamma} = (-1)^{\sigma_n} = -1. \tag{6.9}$$

If $a_{11}a_{33} - a_{31}a_{13} > 0$, by Proposition 6.2, there exists a positive constant D_2, such that when $d_2 \geqslant D_2$, (6.5b) holds and

$$\tilde{\mu}_1(d_2) \in (\mu_k, \mu_{k+1}), \qquad \tilde{\mu}_3(d_2) \in (\mu_n, \mu_{n+1}), \quad k < n. \tag{6.6b}$$

In view of (6.5b) and (6.6b), it follows from (6.8) that

$$\begin{cases} H(\mu_0) = H(0) > 0, \\ H(\mu_i) > 0, \quad 1 \leqslant i \leqslant k, \\ H(\mu_i) < 0, \quad k+1 \leqslant i \leqslant n, \\ H(\mu_{i+1}) > 0, \quad i \geqslant n+1. \end{cases}$$

By the same way, we also have that

$$\mathrm{index}\big(\mathbf{F}(1;\cdot),\tilde{\mathbf{u}}\big) = (-1)^{\gamma} = (-1)^{\sigma_n} = -1.$$

Now, we shall prove that

$$\mathrm{index}\big(\mathbf{F}(0;\cdot),\tilde{\mathbf{u}}\big) = (-1)^0 = 1. \tag{6.10}$$

Fix b_0 such that $b < b_0 < e$ and $a_{11}(b_0) < 0$. Define $b(s) = sb + (1-s)b_0$ for $s \in [0,1]$, and consider problem (4.1), where (d_1, d_2, d_3) and b are replaced by $(\hat{d}_1, \hat{d}_2, \hat{d}_3)$ and $b(s)$, respectively. Precisely, we label this problem as (4.1s), and denote the corresponding non-linear term $\mathbf{G}(\mathbf{u})$ by $\mathbf{G}(s;\mathbf{u})$. As $b \leqslant b(s) < e$ for all $s \in [0,1]$. Similar to the proof of Theorem 5.1 we have that $\tilde{\mathbf{u}}$ is only positive solution of (4.1s) for all $s \in [0,1]$. Same as above, we define

$$\widehat{\mathbf{F}}(s;\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I}-\Delta)^{-1}\big\{\widehat{\mathscr{D}}^{-1}\mathbf{G}(s;\mathbf{u}) + \mathbf{u}\big\} = 0 \quad \text{in } \mathbf{X}^+.$$

Then $\widehat{\mathbf{F}}(1;\cdot) = \mathbf{F}(0;\cdot)$, and $\tilde{\mathbf{u}}$ is the only positive solution of $\widehat{\mathbf{F}}(s;\mathbf{u}) = 0$ for all $s \in [0,1]$. The homotopy invariance of the topological degree asserts that

$$\mathrm{index}\big(\widehat{\mathbf{F}}(1;\cdot),\tilde{\mathbf{u}}\big) = \mathrm{index}\big(\widehat{\mathbf{F}}(0;\cdot),\tilde{\mathbf{u}}\big). \tag{6.11}$$

Since $b(0) = b_0$ and b_0 satisfies $a_{11}(b_0) < 0$, then $\det(\mu_i\widehat{\mathscr{D}} - \mathbf{G}_u(0;\tilde{\mathbf{u}})) > 0$, for all $i \geqslant 1$. Consequently, by Proposition 6.1, $\mathrm{index}(\widehat{\mathbf{F}}(0;\cdot),\tilde{\mathbf{u}}) = (-1)^0 = 1$. Applying $\widehat{\mathbf{F}}(1;\cdot) = \mathbf{F}(0;\cdot)$ and (6.11) we see that (6.10) holds.

By Theorems 4.1 and 4.2, there exists a positive constant c such that, for all $0 \leqslant t \leqslant 1$, the positive solutions of (6.7) satisfy $1/c < u_1, u_2, u_3 < c$. Therefore, $\mathbf{F}(t; \mathbf{u}) \neq 0$ on $\partial\mathcal{B}(c)$ for all $1/c < u_1, u_2, u_3 < c$. By the homotopy invariance of the topological degree,

$$\deg\big(\mathbf{F}(1; \cdot), 0, \mathcal{B}(c)\big) = \deg\big(\mathbf{F}(0; \cdot), 0, \mathcal{B}(c)\big). \tag{6.12}$$

On the other hand, by our assumption, both equations $\mathbf{F}(1; \mathbf{u}) = 0$ and $\mathbf{F}(0; \mathbf{u}) = 0$ have only the positive solution $\tilde{\mathbf{u}}$ in $\mathcal{B}(c)$, and hence, by (6.9) and (6.10),

$$\deg\big(\mathbf{F}(0; \cdot), 0, \mathcal{B}(c)\big) = \operatorname{index}\big(\mathbf{F}(0; \cdot), \tilde{\mathbf{u}}\big) = 1,$$
$$\deg\big(\mathbf{F}(1; \cdot), 0, \mathcal{B}(c)\big) = \operatorname{index}\big(\mathbf{F}(1; \cdot), \tilde{\mathbf{u}}\big) = -1.$$

This contradicts (6.12) and the proof is complete. □

7. Bifurcation

In this section, we discuss the bifurcation of non-constant positive solutions of (4.1). Let the parameters Λ, d_1, d_3 be fixed. We shall only consider the bifurcation with respect to the parameter d_2.

We say that $(\tilde{d}_2; \tilde{\mathbf{u}}) \in (0, \infty) \times \mathbf{X}$ is a bifurcation point of (4.1) if for any $\delta \in (0, \tilde{d}_2)$, there exists $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$ such that (4.1) has a non-constant positive solution. Otherwise, we say that $(\tilde{d}_2; \tilde{\mathbf{u}})$ is a regular point.

In the sequel, we shall denote $S_p = \{\mu_1, \mu_2, \ldots\}$ as the positive spectrum of $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Define

$$\mathcal{N}(d_2) = \big\{\mu > 0 \;\big|\; H(d_2; \mu) = 0\big\}, \quad \text{for } d_2 > 0,$$

where $H(d_2; \mu)$ is introduced by (6.2). Then $\mathcal{N}(d_2)$ contains at most two elements.

Theorem 7.1. *Assume that the parameter Λ satisfies* (1.3). *Let $\tilde{d}_2 > 0$.*

(i) *If $S_p \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, then $(\tilde{d}_2; \tilde{\mathbf{u}})$ is a regular point of* (4.1).
(ii) *Suppose $S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$, and the positive roots of $H(\tilde{d}_2; \mu) = 0$ are simple. If the sum $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$ is odd, then $(\tilde{d}_2; \tilde{\mathbf{u}})$ is a bifurcation point of* (4.1).

Proof. Let $\upsilon(x) = \mathbf{u}(x) - \tilde{\mathbf{u}}$. Then the problem (4.1) is equivalent to

$$\begin{cases} -\Delta\upsilon = \mathscr{D}^{-1}\mathbf{G}(\tilde{\mathbf{u}} + \upsilon), & x \in \Omega, \\ \partial_\nu \upsilon = 0, & x \in \partial\Omega, \end{cases}$$

which, in turn, is equivalent to

$$f(d_2; \upsilon) \triangleq \upsilon - (\mathbf{I} - \Delta)^{-1}\big\{\mathscr{D}^{-1}\mathbf{G}(\tilde{\mathbf{u}} + \upsilon) + \upsilon\big\} = 0 \quad \text{on } \mathbf{X}. \tag{7.1}$$

By direct computation, we have

$$\mathbf{D}_\upsilon f(d_2; 0) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\big(\mathscr{D}^{-1}\mathbf{G}_u(\tilde{\mathbf{u}}) + \mathbf{I}\big),$$

and as in Section 6, for each i, ξ is an eigenvalue of $\mathbf{D}_\upsilon f(d_2; 0)$ on $\mathbf{X}_i$ if and only if $\xi(1 + \mu_i)$ is an eigenvalue of the matrix $H(d_2; \mu_i)$.

(i) If $S_p \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, then $\det H(d_2; \mu_i) \neq 0$ for all i, i.e., 0 is not the eigenvalue of $\mathbf{D}_\upsilon f(\tilde{d}_2; 0)$. This implies that $\mathbf{D}_\upsilon f(\tilde{d}_2; 0)$ is a homeomorphism from $\mathbf{X}$ to itself. The implicit function theorem shows that for all d_2 close to $\tilde{d}_2$, $\upsilon = 0$ is the only solution to $f(d_2; \upsilon) = 0$ in a small neighborhood of the origin, i.e., $(\tilde{d}_2; \tilde{\mathbf{u}})$ is a regular point of (7.1).

(ii) If $S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$, it is easy to show that 0 is a simple eigenvalue of $H(d_2; \mu_i)$ for any i satisfying $\mu_i \in S_p \cap \mathcal{N}(\tilde{d}_2)$. Now, suppose on the contrary that the assertion of the theorem is false. Then there exists $\tilde{d}_2 > 0$ such that the following are true:

(a) $S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$ and $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$ is odd.
(b) There exists $\delta \in (0, \tilde{d}_2)$ such that for every $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$, $\upsilon = 0$ is the only solution to $f(d_2; \upsilon) = 0$ in a neighborhood B_δ of the origin.

Since $f(d_2; \cdot)$ is a compact perturbation of an identity function, in view of (b), the Leray–Schauder degree $\deg(F(d_2; \cdot), B_\delta, 0)$ is well defined and does not depend on $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$. In addition, for those $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$, $\mathbf{D}_\upsilon f(d_2; 0)$ is invertible. Since the positive roots of $H(\tilde{d}_2; \mu) = 0$ are simple, by Proposition 6.1, we have $\deg(f(d_2; \cdot), B_\delta, 0) = (-1)^{\sigma(d_2)}$.

Let

$$\widetilde{H}(d_2; \mu) = d_1 d_2 d_3 H(d_2; \mu).$$

For $\mu_i \in S_p \cap \mathcal{N}(\tilde{d}_2)$, as $\widetilde{H}(\tilde{d}_2; \mu_i) = 0$, a direct computation yields

$$\frac{\partial}{\partial d_2} \widetilde{H}(\tilde{d}_2; \mu_i) = \tilde{d}_2^{-1} a_{12} a_{21} (d_3 \mu_i - a_{33}) < 0.$$

Since $S_p \cap \mathcal{N}(\tilde{d}_2)$ contains at most two elements, there exists $\delta \ll 1$ such that

$$\frac{\partial}{\partial d_2} \widetilde{H}(d_2; \mu_i) < 0,$$

for all $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$ and $\mu_i \in S_p \cap \mathcal{N}(\tilde{d}_2)$. Therefore

$$\widetilde{H}(\tilde{d}_2 - \delta; \mu_i) \widetilde{H}(\tilde{d}_2 + \delta; \mu_i) < 0,$$

and in turn,

$$H(\tilde{d}_2 - \delta; \mu_i) H(\tilde{d}_2 + \delta; \mu_i) < 0, \quad \forall \mu_i \in S_p \cap \mathcal{N}(\tilde{d}_2). \tag{7.2}$$

Since S_p does not have any accumulation point, by taking δ sufficiently small, we may assume that $\mathcal{N}(d_2) \cap S_p = \emptyset$ for all $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2) \cup (\tilde{d}_2, \tilde{d}_2 + \delta]$. Therefore, $\mathbf{D}_\upsilon f(d_2; 0)$ is invertible for all $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2) \cup (\tilde{d}_2, \tilde{d}_2 + \delta]$.

Now, for each i and $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$, $\mathbf{X}_i$ is invariant under $\mathbf{D}_\upsilon f(d_2; 0)$, and the number of eigenvalues with negative real parts of $\mathbf{D}_\upsilon f(d_2; 0)$ on $\mathbf{X}_i$ is the same as that of the matrix $H(d_2; \mu_i)$. If $\mu_i \notin \mathcal{N}(\tilde{d}_2)$, then the number of eigenvalues with negative real parts of $\mathbf{D}_\upsilon f(d_2; 0)$ on $\mathbf{X}_i$ is independent of $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$; whereas if $\mu_i \in \mathcal{N}(\tilde{d}_2)$ then the difference between the number of eigenvalues with negative real parts of $\mathbf{D}_\upsilon f(d_2; 0)$ on $\mathbf{X}_i$ for $d_2 = \tilde{d}_2 - \delta$ and $d_2 = \tilde{d}_2 + \delta$ is 1 by (7.2). Thus, $\sigma(\tilde{d}_2 + \delta) - \sigma(\tilde{d}_2 - \delta)$ is equal to the sum $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$, which is odd. Therefore

$$\deg\big(f(\tilde{d}_2 - \delta, \cdot), B_\delta, 0\big) \neq \deg\big(f(\tilde{d}_2 + \delta, \cdot), B_\delta, 0\big),$$

and we have a contradiction. This shows that $(\tilde{d}_2; \tilde{\mathbf{u}})$ is a bifurcation point of (7.1). $\square$

Theorem 7.2. *Assume that the parameter Λ satisfies* (1.3), $S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$, *and the positive roots of $H(\tilde{d}_2; \mu) = 0$ are simple. If the sum $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$ is odd, then there exists an interval $(\alpha, \beta) \subset \mathbf{R}^+$ such that for every $d_2 \in (\alpha, \beta)$, the problem* (4.1) *admits a non-constant positive solution $\mathbf{u} = \mathbf{u}(d_2)$. Moreover, one of the following holds*:

(i) $\tilde{d}_2 = \alpha < \beta < \infty$ *and* $S_p \cap \mathcal{N}(\beta) \neq \emptyset$;
(ii) $0 < \alpha < \beta < \tilde{d}_2$ *and* $S_p \cap \mathcal{N}(\alpha) \neq \emptyset$;
(iii) $\mathbf{u}(\alpha) = \tilde{\mathbf{u}}$ *or* $\mathbf{u}(\beta) = \tilde{\mathbf{u}}$;
(iv) $(\alpha, \beta) = (\tilde{d}_2, \infty)$;
(v) $(\alpha, \beta) = (0, \tilde{d}_2)$.

Proof. Let $\Gamma = \{d_2 > 0 \mid S_p \cap \mathcal{N}(d_2) \neq \emptyset\}$, $S = \text{closure}\{(d_2, \mathbf{u}) \in \mathbf{R}^+ \times \mathbf{X} \mid \mathbf{u} > 0,\ \mathbf{u} \neq \tilde{\mathbf{u}},\ \mathbf{u} \text{ solves (4.1)}\}$.

In view of the estimates (4.2) and (4.4), following the arguments of [14] or [15], and incorporating the calculation of the degree $\deg(f(d_2; \cdot), B_\delta, 0)$ that we presented in the proof of Theorem 7.1, we can conclude that S contains a maximal connected subset $\mathcal{C}$ which emanates from $(\tilde{d}_2; \tilde{\mathbf{u}})$ and

(1) $\mathcal{C}$ meets $\Gamma \times \{\tilde{\mathbf{u}}\}$ at a point $(d_2; \tilde{\mathbf{u}})$ with $d_2 \neq \tilde{d}_2$; or
(2) $\mathcal{C}$ meets $\{d_2 > 0\} \times \{\tilde{\mathbf{u}}\}$ at a point $(d_2; \tilde{\mathbf{u}})$ with $d_2 \neq \tilde{d}_2$; or
(3) $\mathcal{C}$ is non-compact in $(0, \infty) \times \mathbf{X}$.

Corresponding to the case (1), either the assertion (i) or the assertion (ii) of the theorem holds. If (2) happens, then (iii) holds. Finally, if (3) holds, then, applying the estimates (4.2) and (4.4), we can easily show that either (iv) or (v) holds. This completes the proof. □

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